

Best Approximation by Closed Sets in Banach Spaces

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1. INTRODUCTION

Let K be a nonempty subset in a (real) Banach space X . For each $x \in X$, we say that $y \in K$ is a *best approximation* to x from K if

$$\|x - y\| = \inf\{\|x - z\| : z \in K\}.$$

The set K is called *proximal (Chebyshev)* if every point $x \in X$ has a (unique) best approximation from K . It is easy to see that every closed convex set K in a reflexive space X is proximal. In addition, if the norm is strictly convex, then K is Chebyshev. However, if X is not assumed reflexive or K is not assumed convex, then the above result is false in general. In [7], Stečkin introduced the concept of almost Chebyshev. A set K is called *almost Chebyshev* if the set of x in X such that K fails to have unique best approximation to x is a first category subset of X . He proved that if X is a uniformly convex Banach space, then every closed subset is almost Chebyshev. By using this concept, Garkavi [4] showed that for any reflexive subspace F in a separable Banach space, there exists a (in fact, many) subspace G which is B -isomorphic to F and is almost Chebyshev. The author [6] showed that if X is a separable Banach space which is locally uniformly convex or possesses the Radon–Nikodym property, then “almost all” closed subspaces are almost Chebyshev. In [3], Edelstein proved that if X has the Radon–Nikodym property, then for any bounded closed convex subset K , the set of x in X which admit best approximations from K is a weakly dense subset in X .

In this paper, we generalize Stečkin’s result to a wider class of Banach spaces. A Banach space is called a *U-space* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|(x + y)/2\| > 1 - \delta$, $\|(x^* + y^*)/2\| > 1 - \epsilon$, where x^* and y^* are norm 1 support functionals of the closed unit ball of X at x, y , respectively. We show that this class of spaces is self-dual, it contains all uniformly convex spaces,

uniformly Fréchet differentiable spaces and is contained in the class of uniformly nonsquare Banach spaces. Our main result is: Every closed subset in a locally uniformly convex U -space is almost Chebyshev.

In Section 2, we obtain some basic properties of the U -spaces. We prove the main results in Section 3.

2. U -SPACES

Throughout, we will use the following notation:

- X real Banach spaces.
- X^* dual of the Banach space X .
- $B(x, r)$ the set of points whose distance to x is less than or equal to r .
- B_r $B(x, r)$ with $x = 0$.
- S_r the set of points with norm equal to r .
- ∇_x the set of norm 1 support functionals of $S_{\|x\|}$ at x .

For each $x \in S_1$, $x^* \in \nabla_x$, and for $1 > r > 0$, $\delta > 0$, we let

$$\begin{aligned} N_r(x, \delta) &= \overline{B_1 \setminus B(-rx, 1 + r - \delta)}, \\ M_r(x, \delta) &= \overline{B(rx, 1 - r + \delta) \setminus B_1}, \\ d(x^*, N_r(x, \delta)) &= \sup\{1 - x^*(y) : y \in N_r(x, \delta)\}, \\ d(x^*, M_r(x, \delta)) &= \sup\{1 + \delta - x^*(y) : y \in M_r(x, \delta)\}. \end{aligned}$$

The following proposition is the motivation of the definition of U -spaces and its geometric characterization will be used in the next section (Lemma 3.1, Proposition 3.2).

PROPOSITION 2.1. *Let X be a Banach space, let $x \in S_1$ and let $x^* \in \nabla_x$. Then the following conditions are equivalent:*

- (i) *For any $\epsilon > 0$, there exists $\delta > 0$ (depends on x, x^*, ϵ) such that for any $y \in S_1$ with $\|(x + y)/2\| > 1 - \delta$, $x^*(y) > 1 - \epsilon$.*
- (ii) $\lim_{\delta \rightarrow 0} d(x^*, N_r(x, \delta)) = 0$ for any $1 > r > 0$.
- (iii) $\lim_{\delta \rightarrow 0} d(x^*, M_r(x, \delta)) = 0$ for any $1 > r > 0$.

Proof. The equivalence of (ii), (iii) follows from the fact that $N_r(x, \delta)$ is a homothetic translation of $M_r(x, \delta)$ and vice versa. To prove (i) implies (ii), suppose there exists $1 > r > 0$ such that

$$\lim_{\delta \rightarrow 0} d(x^*, N_r(x, \delta)) > 2\epsilon \quad \text{for some } \epsilon > 0.$$

Let $\delta_0 > 0$ be a number satisfying

$$\|(x + y)/2\| > 1 - \delta_0, \quad y \in S_1 \Rightarrow x^*(y) > 1 - \epsilon. \quad (*)$$

For $\delta = \min\{r\delta_0/2, \epsilon/2\}$, choose $y \in N_r(x, \delta)$ such that $1 - x^*(y) > 2\epsilon$, i.e., $x^*(y) < 1 - 2\epsilon$. Note that $\|y + rx\| \geq 1 + r - \delta$, hence $\|y\| \geq 1 - \delta$; also note that $\|y\| \leq 1$. Let $y_0 = y/\|y\|$, then $\|y - y_0\| < 2\delta$ and

$$x^*(y_0) < 1 - 2\epsilon + 2\delta \leq 1 - 2\epsilon + \epsilon = 1 - \epsilon.$$

It follows from (*) that

$$\|(x + y_0)/2\| \leq 1 - \delta_0.$$

Thus

$$\begin{aligned} \|y + rx\| &\leq \|y - y_0\| + \|r(y_0 + x) - (1 - r)y_0\| \\ &\leq 2\delta + 2r(1 - \delta_0) + (1 - r) \\ &< 1 + r - \delta. \end{aligned}$$

This contradicts that y is in $N_r(x, \delta)$.

To prove the sufficiency, suppose that (i) were not true, we can find $\epsilon > 0$ such that for any $\delta > 0$, there exists $y \in S_1$ with

$$\|(x + y)/2\| > 1 - \delta, \quad \text{but } x^*(y) \leq 1 - \epsilon. \quad (**)$$

By (ii), there exists δ_0 such that $d(x^*, N_r(x, \delta_0)) < \epsilon$. Consider $\delta = \delta_0/2$, there exists $y \in S_1$ satisfies (**), hence $y \notin N_r(x, \delta_0)$ and $\|y + rx\| < 1 + r - \delta_0$. Now,

$$\begin{aligned} \left\| \frac{x + y}{2} \right\| &= \left\| \frac{(1 - r)x}{2} + \frac{rx + y}{2} \right\| \\ &< \frac{1 - r}{2} + \frac{1 + r - \delta_0}{2} \\ &\leq 1 - \frac{\delta_0}{2} = 1 - \delta. \end{aligned}$$

This contradicts the choice of y .

DEFINITION 2.2. A Banach space X is called a U -space if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in S_1$ with $\|(x + y)/2\| > 1 - \delta$, $x^*(y) > 1 - \epsilon$ for any $x^* \in \nabla_x$.

It is clear that in the above definition we can assume $x, y \in B_1$ instead of $x, y \in S_1$. It also follows easily from the definition that X is a U -space if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in S_1$ with $\|(x + y)/2\| > 1 - \delta$, $\|(x^* + y^*)/2\| > 1 - \epsilon$ for all $x^* \in \nabla_x, y^* \in \nabla_y$.

In the rest of this section, we will give some classification of the U -spaces. A Banach space is called *uniformly nonsquare* [2, 5] if there exists $0 < a < 1$, satisfies for any x, y in B_1 , either $\|(x + y)/2\| \leq a$ or $\|(x - y)/2\| \leq a$. Following directly from the definitions, we can show that every U -space is uniformly nonsquare. Also, it is easy to see that a uniformly nonsquare space is not necessarily a U -space. (The two-dimensional Banach space with the norm generated by a hexagon will be an example.) In [5], it is proved that every uniformly nonsquare space is reflexive. Hence, we have

COROLLARY 2.3. *Every U -space is reflexive.*

THEOREM 2.4. *Let X be a Banach space, then X is a U -space if and only if X^* is a U -space.*

Proof. By the above, it suffices to show that X^* is a U -space implies X is a U -space. For any $\epsilon > 0$, let $\delta > 0$ be a number satisfies

$$\left\| \frac{x^* + y^*}{2} \right\| > 1 - \delta, \quad x^*, y^* \in S_1^* \Rightarrow x^*(y) > 1 - \epsilon, \quad y \in \nabla_{y^*}. \quad (*)$$

Let $x, y \in S_1$ satisfy $\|(x + y)/2\| > 1 - \delta$. For $z^* \in \nabla_{(x+y)/2}$, $z^*((x + y)/2) > 1 - \delta$, hence

$$z^*(x) > 1 - 2\delta, \quad z^*(y) > 1 - 2\delta.$$

It follows that

$$\left(\frac{z^* + x^*}{2} \right) (x) > 1 - \delta, \quad \left(\frac{z^* + y^*}{2} \right) (y) > 1 - \delta$$

and

$$\left\| \frac{z^* + x^*}{2} \right\| > 1 - \delta, \quad \left\| \frac{z^* + y^*}{2} \right\| > 1 - \delta.$$

Choose $z \in \nabla_{z^*}$; by (*), we have

$$x^*(z) > 1 - \epsilon, \quad y^*(z) > 1 - \epsilon.$$

This implies $\|(x^* + y^*)/2\| > 1 - \epsilon$ and completes the proof.

A Banach space is called *locally uniformly convex* if for any $x \in S_1$ and for any $\epsilon > 0$, there exists $\delta > 0$ (depends on ϵ, x) such that for any $y \in S_1$ with $\|x - y\| > \epsilon$, $\|(x + y)/2\| > 1 - \delta$. It is called *uniformly convex* if the δ above can be chosen independent of $x \in S_1$. A Banach space is called *uniformly Fréchet differentiable* if $\lim_{y \rightarrow 0} (\|x + y\| - \|x\|)/\|y\|$ exists for all $x \in S_1$ and the limit is independent of x . It is well known [2] that X is uniformly convex if and only if X^* is uniformly Fréchet differentiable.

COROLLARY 2.5. *Uniformly convex spaces and uniformly Fréchet differentiable spaces are U-spaces.*

3. BEST APPROXIMATION

Let K be a closed subset in a Banach space X , we define the distance function from x to K as

$$r(x) = \inf\{\|x - z\| : z \in K\}.$$

It is clear that $|r(x) - r(y)| \leq \|x - y\|$ for all $x, y \in X$. For $\delta > 0$, we let

$$\begin{aligned} K_\delta(x) &= B(x, r(x) + \delta) \cap K, \\ K_\delta^*(x) &= \{z^* : z^* \in \nabla_{z-x}, \text{dist}(z, K_\delta(x)) < \delta\}, \\ d_\delta(x) &= \sup\{z^*(y_1 - y_2) : y_1, y_2 \in K_\delta(x), z^* \in K_\delta^*(x)\}, \end{aligned}$$

and

$$d(x) = \lim_{\delta \rightarrow 0} d_\delta(x).$$

We remark that $K_\delta(x)$, $K_\delta^*(x)$ are decreasing as $\delta \rightarrow 0$, hence $d_\delta(x)$ is decreasing and the limit exists.

Let X be a U -space, it is clear that the two limits $\lim_{\delta \rightarrow 0} d(x^*, N_r(x, \delta))$ and $\lim_{\delta \rightarrow 0} d(x^*, M_r(x, \delta))$ in Proposition 2.1 converge to 0 uniformly for $x \in S_1$ and $x^* \in \nabla_x$. In the following, we need a slightly stronger result.

LEMMA 3.1. *Let X be a U -space. Then for $\epsilon > 0$, $1 > r > 0$, there exists $\delta > 0$ satisfies*

$$|z^*(y_1 - y_2)| < \epsilon$$

for all $y_1, y_2, z \in M_r(x, \delta)$, $z^* \in \nabla_{z-rx}$, $x \in S_1$.

Proof. Note that $M_r(x, \delta)$ and $N_r(x, \delta)$ are homothetic translations of each other, it will be more convenient to prove: for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2, z \in N_r(x, \delta)$, $z^* \in \nabla_z$, $x \in S_1$,

$$|z^*(y_1 - y_2)| < \epsilon.$$

Since X is a U -space, we can find $\delta_1 > 0$ such that

$$\left\| \frac{y_1 + y_2}{2} \right\| > 1 - \delta_1, \quad y_1, y_2 \in B_1 \Rightarrow y_1^*(y_2) > 1 - (\epsilon/2), \quad y_1^* \in \nabla_{y_1}. \quad (*)$$

As $\lim_{\delta \rightarrow 0} d(x^*, N_r(x, \delta)) = 0$ uniformly for $x \in S_1$, $x^* \in \nabla_x$, there exists $\delta < \delta_1$ (independent of $x \in S_1$, $x^* \in \nabla_x$) such that for $y \in N_r(x, \delta)$, $x^*(y) > 1 - \delta_1$. Hence for $y_1, y_2, z \in N_r(x, \delta)$, we have

$$x^*((z + y_i)/2) > 1 - \delta_1, \quad i = 1, 2.$$

This implies

$$\|(z + y_i)/2\| > 1 - \delta_1, \quad i = 1, 2.$$

By (*), $z^*(y_i) > 1 - (\epsilon/2)$, for $z^* \in \nabla_z$, $i = 1, 2$, hence $|z^*(y_1 - y_2)| < \epsilon$.

PROPOSITION 3.2. *Let X be a U -space and let K be a closed subset in X , then the set $\{x: d(x) = 0\}$ is a dense G_δ in X .*

Proof. We will show that for each n , the set $F_n = \{x: d(x) \geq (1/n)\}$ is closed and contains no interior. The Baire theorem will imply that $\bigcup_{n=1}^{\infty} F_n$ is a nowhere dense set and hence $\{x: d(x) = 0\}$ is a dense G_δ .

To show that F_n is closed, let $x_0 \notin F_n$. Without loss of generality, we assume that $x_0 = 0$. There exist $\delta_0 > 0$, such that

$$z^*(y_1 - y_2) < (1/n) \quad \forall y_1, y_2 \in K_{\delta_0}(0), \quad z^* \in K_{\delta_0}^*(0).$$

Choose $\delta_1 = (\delta_0/3)$, for $\|x\| < \delta_1$, we have

$$(i) \quad r(x) + \delta_1 \leq r(0) + 2\delta_1, \text{ hence } K_{\delta_1}(x) \subseteq K_{\delta_0}(0).$$

$$(ii) \quad z^* \in K_{\delta_1}^*(x) \Rightarrow z^* \in \nabla_{z-x}, \text{ where } \text{dist}(z, K_{\delta_1}(x)) < \delta_1, \\ \Rightarrow z^* \in \nabla_{z-x}, \text{ where } \text{dist}(z - x, K_{\delta_0}(0)) < \delta_0, \\ \Rightarrow z^* \in K_{\delta_0}^*(0),$$

i.e., $K_{\delta_1}^*(x) \subseteq K_{\delta_0}^*(0)$.

It follows that for $\|x\| < \delta_1$, we have

$$z^*(y_1 - y_2) < (1/n) \quad \forall y_1, y_2 \in K_\delta(x), \quad z^* \in K_\delta^*(x);$$

hence $d_{\delta_1}(x) < 1/n$ and $B_{\delta_1} \cap F_n = \emptyset$. This completes the proof that F_n is closed.

Assume that F_n had nonvoid interior. Without loss of generality, let $B_r \subseteq F_n^0$ ($1 > r > 0$) and $r(0) = 1$. By Lemma 3.1, there exists $\delta > 0$ satisfies

$$z^*(y_1 - y_2) < 1/n \quad \forall y_1, y_2, z \in M_r(x, \delta), \quad z^* \in \nabla_{z-x}, \quad x \in S_1. \quad (*)$$

Let $\delta_0 = \delta/3$, choose $x_0 \in B_{1+\delta_0} \cap K$. Let $x_1 = x_0/\|x_0\|$, $x_r = rx_0$. Then

$$K_{\delta_0}(x_r) \subseteq M_r(x_1, 2\delta_0)$$

and for z such that $\text{dist}(z, K_{\delta_0}(x_r)) < \delta_0$, let $z' = \lambda z + (1 - \lambda)x_r$ with $\lambda > 0$ and $\|z'\| = 1$. It is easy to show that

$$z' \in M_r(x_1, 3\delta_0) = M_r(x_1, \delta).$$

Thus (*) implies

$$z^*(y_1 - y_2) < 1/n \quad \forall y_1, y_2 \in K_{\delta_0}(x_r), \quad z^* \in K_{\delta_0}^*(x_r);$$

i.e., $d(x_r) < 1/n$. This contradicts that $x_r \in F_n^0$.

DEFINITION 3.3. A subset K in a Banach space is called almost Chebyshev if the set of $x \in X$ which fails to have unique best approximation from K to x is a first category subset of X .

Recall that a locally uniformly convex space has the following property: If $x_n \xrightarrow{w} x$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \xrightarrow{\|\cdot\|} x$.

THEOREM 3.4. Let X be a locally uniformly convex U -space, every closed subset in X is almost Chebyshev.

Proof. By Proposition 3.2, we know that the set $G = \{x: d(x) = 0\}$ is a dense G_δ . Let $x \in G$, choose $y_n \in B(x, r(x) + (1/n)) \cap K$. Without loss of generality, by the reflexivity of X , we may assume that $\{y_n\}$ converges to y weakly. Since $d(x) = 0$ and $\lim_{m \rightarrow \infty} y_n^*(y_m - x) = y_n^*(y - x)$, $y_n^* \in \nabla_{y_n - x}$, we can show that for any $\epsilon > 0$, $y \in \{z: y_n^*(z - x) \geq r(x) - \epsilon\}$ for some n . This implies $\|y - x\| = r(x)$ and $\|y_n - x\| \rightarrow \|y - x\|$. Since X is locally uniformly convex, by the above remark, $y_n \xrightarrow{\|\cdot\|} x$. That K is closed implies $y \in K$ and

$$\|x - y\| = r(x) = \inf\{\|x - z\|: z \in K\}.$$

Hence, every point $x \in G$ is a best approximation from K . It is proved in [7] that under the same assumption, the set in X which has not more than one best approximation from K is also a dense G_δ . Together with what we proved above, we conclude that K is almost Chebyshev.

Remarks. (1) By a renorming theorem of Asplund [1], we can construct a locally uniformly convex, uniformly Fréchet differentiable space which is not uniformly convex; hence, Theorem 3.5 generalizes the result of Stečkin.

(2) We do not know whether Theorem 3.4 will hold for reflexive locally uniformly convex spaces. It is interesting to know whether similar result holds for U -spaces (In this case, we have to give up the requirement of uniqueness in the definition of almost Chebyshev subsets).

(3) Edelstein [3] gave an example that the above theorem may not hold in separable, strictly convex reflexive Banach spaces.

Note added in proof. Recently the author proved that Theorem 3.4 holds for reflexive locally uniformly convex spaces.

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